Second semester 2012-2013
Midsemestral exam
Algebraic Number Theory
B.Math.(Hons.) IIIrd year
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Q 1. Let A be a Dedekind domain. If I is any non-zero ideal and $a \in I$ a non-zero element, prove that there exists $b \in I$ such that $I = (a, b)$.

OR

Define a fractional ideal I in an integral domain A. If I is a fractional ideal such that there exists a fractional ideal J with $IJ = A$, then prove $J = \{x \in K : xI \subset A\}$.

OR

If A is a Dedekind domain, prove that the group $\mathcal{P}$ of principal fractional ideals is isomorphic to K^*/A^* where K is the quotient field of A and A^* denotes the subgroup of units in A.

Q 2. Let d be a square-free integer and let $K = \mathbf{Q}(\sqrt{d})$. If p is an odd prime, derive the decomposition of pO_K using the Kummer-Dedekind criterion.

OR

If L/K be a Galois extension of algebraic number fields, and P is a non-zero prime ideal of O_K, prove that the Galois group permutes the prime ideals lying over P transitively.

Q 3. Let K denote the number field generated by a primitive p^n-th root of unity where p is an odd prime. Show that the discriminant of K is $(-1)^{(p-1)/2}p^{(n(p-1)-1)p^{n-1}}$.

OR

Let A be an integrally closed domain. Let $f \in A[X]$ be a monic polynomial so that $f = gh$ where $g, h \in K[X]$ are monic with K, the quotient field of A. Prove that $g, h \in A[X]$.

Q 4. Let K be an algebraic number field. If p is a prime number and $pO_K = P_1^{e_1} P_2^{e_2} \cdots P_g^{e_g}$ where P_i are prime ideals of O_K, prove that the set of prime ideals of O_K lying over p is precisely $\{P_1, \cdots, P_g\}$.

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Q 5. If $\alpha \in O_K$ for an algebraic number field K of degree n, then prove that the norm of the ideal αO_K is $|N_{K/\mathbf{Q}}(\alpha)|$.

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Q 6. if f is the minimal polynomial of α, prove that the discriminant of $\{1, \alpha, \cdots, \alpha^{n-1}\}$ is $(-1)^{n(n-1)/2} N_{K/\mathbf{Q}}(f'(\alpha))$.

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Q 7. Let α be a root of $f(X) = X^5 - X - 1$ and let $K = \mathbf{Q}(\alpha)$. The polynomial f is irreducible and the discriminant of K is 19×151 - assume these. Show that the ideal $(19, \alpha + 6)^2$ divides $19O_K$.

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Q 8. Consider $L := \mathbf{Q}(\zeta_{p^2})$, where ζ_{p^2} is a primitive p^2-th root of unity with p an odd prime. Let K be the unique subfield of L which has degree p over $\mathbf{Q}$. Show that 2 splits completely in K if and only if $2^{p-1} \equiv 1 \bmod p^2$.

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Q 9. Assuming that 3 does not divide the class number of $\mathbf{Q}(\sqrt{-5})$, deduce that $x^2 + 5 = y^3$ does not have any integer solutions.